1 Angular Momentum

1.1 Commutator Relations

Quantum Mechanical operators $\mathbf{L}_x$, $\mathbf{L}_y$, $\mathbf{L}_z$ measure angular momentum if they satisfy the commutator relations

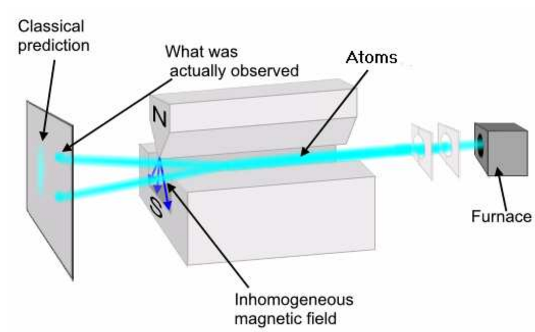
$$[\mathbf{L}_x, \mathbf{L}_y] = i\hbar\, \mathbf{L}_z,$$
$$[\mathbf{L}_y, \mathbf{L}_z] = i\hbar\, \mathbf{L}_x,$$
$$[\mathbf{L}_z, \mathbf{L}_x] = i\hbar\, \mathbf{L}_y$$

2 Intrinsic Spin

Observation has shown that particles have "intrinsic" spin (units of angular momentum).

2.1 Spin ½ Particles

Experiments (e.g. Stern-Gerlach) have shown that electrons, some atoms and other sub-atomic particles have an intrinsic "2-valued" angular momentum.



2.1.1 Pauli Spin Matrices

There are 3 distinct orientations in space (denoted x, y and z), and there are 2 distinct values (denoted "up" and "down")

Pauli Matrices provide a model for the measurement of angular momentum in each of the 3 standard spatial orientations. Standard representation and resulting eigenvectors are shown below.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \left|Up(x)\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda = 1 \qquad \left|Down(x)\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda = -1$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \left|Up(y)\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}, \lambda = 1 \qquad \left|Down(y)\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}, \lambda = -1$$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad |Up(z)\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = 1$ $\quad |Down(z)\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda = -1$

2.1.1.1 Algebraic properties

$\sigma_x{}^2 = \sigma_y{}^2 = \sigma_z{}^2 = 1$
$\sigma_x.\ \sigma_y = i.\sigma_z$
$\sigma_y.\ \sigma_z = i.\sigma_x$
$\sigma_z.\ \sigma_x = i.\sigma_y$
$[\sigma_x, \sigma_y] = 2i.\sigma_z$
$[\sigma_y, \sigma_z] = 2i.\sigma_x$
$[\sigma_z, \sigma_x] = 2i.\sigma_y$

2.1.2 Relationship between Pauli matrices and spin ½

Put $\mathbf{L}_x = \lambda.\sigma_x$, $\mathbf{L}_y = \lambda.\sigma_y$, $\mathbf{L}_z = \lambda.\sigma_z$, then

$[\mathbf{L}_x, \mathbf{L}_y] = [\lambda.\sigma_x, \lambda.\sigma_y] = |\lambda|^2 [\sigma_x, \sigma_x] = |\lambda|^2.2.i.\sigma_z$
But $[\mathbf{L}_x, \mathbf{L}_y] = i\hbar\ \mathbf{L}_z = i\hbar\ (\lambda.\sigma_z)$
$\Rightarrow \lambda = \hbar/2$

$\mathbf{L}_x = \hbar/2.\sigma_x$, $\mathbf{L}_y = \hbar/2.\sigma_y$, $\mathbf{L}_z = \hbar/2.\sigma_z$

σ_x, σ_y and σ_z have known eigenvectors, with eigenvectors $\lambda = \pm 1$.
$\mathbf{L}_x$, $\mathbf{L}_y$ and $\mathbf{L}_x$ have the same eigenvectors, with eigenvectors $\lambda = \pm$ ½.ħ.

I.e. the particle spin of ± ½.ħ is the result of the commutator relations.

2.1.3 Quarterions

These are closely related to Pauli matrices and sometime appear in the literature in their place. Quarternions are algebraic extensions to Complex numbers.

$$i^2 = j^2 = k^2 = -1.$$
$$i.j.k = 1$$
$$i.j = k$$
$$j.k = i$$
$$k.i = j$$

2.1.3.1 Quarternions - Preferred representation

$\mathbf{i} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i.\sigma_x$ $\quad \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i.\sigma_y$ $\quad \mathbf{k} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i.\sigma_z$

2.2 Spin 1 Particles

Spin 1 particles have "3-valued" angular momentum. Standard representations are shown below for the angular momentum operator and eigenvectors are shown below.

$$\mathbf{L}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left| L_x = 1 \right\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \lambda = \sqrt{2}$$

$$\left| L_x = 0 \right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \lambda = 0$$

$$\left| L_x = -1 \right\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \lambda = -\sqrt{2}$$

$$\mathbf{L}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$\left| L_y = -1 \right\rangle = \frac{1}{2}\begin{bmatrix} -i \\ \sqrt{2} \\ i \end{bmatrix}, \lambda = \sqrt{2}$$

$$\left| L_y = 0 \right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \lambda = 0$$

$$\left| L_y = -1 \right\rangle = \frac{1}{2}\begin{bmatrix} i \\ -\sqrt{2} \\ -i \end{bmatrix}, \lambda = -\sqrt{2}$$

$$\mathbf{L}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\left|L_z = 1\right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \lambda = 1$$

$$\left|L_z = 0\right\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \lambda = 0$$

$$\left|L_z = -1\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \lambda = -1$$

2.3 Field Angular Momentum (Orbital Angular Momentum)

If a system is described by state $\Psi(x, y, z)$, then.

$$\mathbf{L}_x = i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\mathbf{L}_y = i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\mathbf{L}_z = i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

2.4 Algebraic Properties of Angular Momentum

Define total angular momentum operator $\mathbf{L}^2 = \mathbf{L}_x^2 + \mathbf{L}_y^2 + \mathbf{L}_z^2$, then

$$[\mathbf{L}^2, \mathbf{L}_x] = [\mathbf{L}^2, \mathbf{L}_y] = [\mathbf{L}^2, \mathbf{L}_z]$$

$\mathbf{L}^2$ and $\mathbf{L}_z$ commute so $\mathbf{L}^2$ and $\mathbf{L}_z$ are simultaneously measurable. Define $| jm >$ as the state where

$$\mathbf{L}^2 \,| jm > = j \,| jm > \text{ and } \mathbf{L}_z \,| jm > = m \,| jm >$$

Define "ladder operators" $\mathbf{L}_+ = \mathbf{L}_x + i.\mathbf{L}_y$ and $\mathbf{L}_- = \mathbf{L}_x - i.\mathbf{L}_y$, then

$$[\mathbf{L}^2, \mathbf{L}_+] = [\mathbf{L}^2, \mathbf{L}_-] = 0$$

$\mathbf{L}_+$ and $\mathbf{L}_-$ commute with $\mathbf{L}^2$ so are simultaneously measureable.

$$\mathbf{L}_+ \,| jm > = | jm_+ > \text{ and } \mathbf{L}_- \,| jm > = | jm_- > \text{ for some } m_+, m_-$$

How does a measurement of $L_\pm$ affect the z-momentum?

$$\begin{aligned} \mathbf{L}_z \mathbf{L}_\pm \,| jm > &= [\mathbf{L}_z, \mathbf{L}_\pm] \,| jm > + \mathbf{L}_\pm \mathbf{L}_z \,| jm > \\ &= (m \pm \hbar) \,| jm > \end{aligned}$$

Note $-j < m < j$. The choice of z direction is arbitrary. If m is a possible value for intrinsic angular momentum in some direction, then so is (m + ħ) provided $-j < m + ħ < j$.

Similarly (m – ħ) is a possible value provided $-j < m - ħ < j$.
In the case of quantised angular momentum, possible values are shown in the pyramid below.

0
-½ ħ ½ ħ
-ħ 0 ħ
-3/2 ħ ½ ħ ½ ħ -3/2 ħ
-2ħ -ħ 0 ħ 2ħ

3 Into the 4th Dimension

3.1 The Dirac Equation

Relativity implies that for a single particle:

$$\boldsymbol{p}^2 - m^2c^2 = 0$$

where $\boldsymbol{p}$ is the 4-momentum. The Quantum Mechanical version of this is (Klein-Gordon)

$$(\partial_\mu{}^2 - m^2c^2)\Psi = 0$$

Assume that the particle waveform can be factored into separate 1st order equations. I.e.

$$(\gamma_\mu.\partial_\mu - imc)\ (\gamma_\mu.\partial_\mu + imc)\Psi = 0$$

where γ_μ. (μ=0,1,2,3) are constants.

$$(\gamma_\mu.\partial_\mu - imc)(\gamma_\tau.\partial_\tau + imc)\Psi = 0$$
$$\Rightarrow (\gamma_\mu.\gamma_\tau.\partial_\mu\partial_\tau - m^2c^2)\Psi = 0$$

This reduces to the Klein-Gordon equation if

$$\gamma_\mu\,\gamma_\tau + \gamma_\tau\,\gamma_\mu = 2\delta_{\mu\tau} \qquad \text{for all } \mu,\tau = 0,1,2,3$$

γ_τ cannot be Real or Complex numbers.

Possible representations of γ_k, γ (4x4 matrices):

$$\gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \qquad \gamma_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix} \text{ for } k = 1,2,3$$

3.2 4-Dimensional Pauli Matrices

The Pauli spin matrices are a canonical representation of a system with three conjugate 2-valued properties.

What would a system with 4 conjugate 2-valued property, denoted by **K**, look like?

We can use the existing Pauli representation for the first three 2-valued properties and look for a representation for the fourth.

Let $\mathbf{\Psi} = \begin{bmatrix} a \\ b \end{bmatrix}$ be an eigenvector of K, then P($\mathbf{\Psi}$ | $\mathbf{e}_i$) for all 6 Pauli eigenvectors.

$$\begin{bmatrix} a \\ b \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a = \frac{1}{\sqrt{2}} \Rightarrow |a| = \frac{1}{\sqrt{2}} \quad \text{... (1)}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b = \frac{1}{\sqrt{2}} \Rightarrow |a| = \frac{1}{\sqrt{2}} \quad \text{... (2)}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} . \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(a+b) = \frac{1}{\sqrt{2}} \Rightarrow |a+b| = 1 \quad \text{... (3)}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} . \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(a-b) = \frac{1}{\sqrt{2}} \Rightarrow |a-b| = 1 \quad \text{... (4)}$$

$\mathbf{\Psi} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm j \end{bmatrix}$ would be a solution for equations (1) ... (4) where j is a unit quarternion.

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm j \end{bmatrix} . \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix} = \frac{1}{2}(1 + i.j) = \frac{1}{\sqrt{2}} \Rightarrow |1 + i.j| = |1 + k| = \sqrt{2} \quad \text{... (5),(6)}$$

Note that i, j, k must be introduced together since if $i.j \neq i$ then

$$i.j - i = 0 \Rightarrow i.(j-i) = 0 \Rightarrow j = i$$

For symmetry purposes, choose eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm j \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm k \end{bmatrix}$$

What would the operators look like? Denote the 4 dimensional operators $\mathbf{K}_0$, $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$.

We know the standard representation for **K**1 (standard Pauli matrix) is $\mathbf{K}_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. By symmetry, the complete representation is

$$\mathbf{K}_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{K}_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}$$

Recall the standard representation

$$\boldsymbol{i} = i.\sigma_x , \boldsymbol{j} = i.\sigma_y , \boldsymbol{k} = i.\sigma_z$$

This implies

$$\mathbf{K}_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{K}_1 = \begin{bmatrix} 0 & -i.\sigma_x \\ i.\sigma_x & 0 \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} 0 & -i.\sigma_y \\ i.\sigma_y & 0 \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} 0 & -i.\sigma_z \\ i.\sigma_z & 0 \end{bmatrix}$$

I.e.

$$\mathbf{K}_0 = \gamma_0,$$
$$\mathbf{K}_1 = -i.\gamma_1,$$
$$\mathbf{K}_2 = -i.\gamma_2,$$
$$\mathbf{K}_3 = -i.\gamma_3$$

where γ_α are the Dirac matrices.

A system of 4 conjugate 2-valued properties is naturally described by 4 vectors and 4 dimensional Dirac matrices! And Relativity + Dirac matrices leads to anti-matter!